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# Bipartite graphs containing every possible pair of cycles

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**Abstract**

Let $G=(V_1,V_2;E)$ be a bipartite graph with $|V_1|=|V_2|=n\geqslant 4$. Suppose that $d(x)+d(y)\geqslant n+2$ for all $x\in V_1$ and $y\in V_2$. Then $G$ contains two vertex-disjoint cycles of lengths $2s$ and $2t$, respectively, for any two positive integers $s$ and $t$ with $s\geqslant 2$, $t\geqslant 2$ and $s+t\leqslant n$. We also propose a conjecture. © 1999 Elsevier Science B.V. All rights reserved.

## 1. Introduction

We discuss only finite simple graphs. We use Bollobás [2] for terminology and notation except as indicated. It is well known [6] that if a graph $G$ of order $n\geqslant 3$ has minimum degree at least $n/2$ then $G$ is hamiltonian. Furthermore, Bondy [3] showed that $G$ is also pancyclic, i.e., $G$ contains a cycle of length $i$ for each $i$, $3\leqslant i\leqslant n$ unless $n$ is even and $G$ is isomorphic to the complete bipartite graph $K_{n/2,n/2}$. So, when the minimum degree of $G$ is at least $\lceil (n+1)/2\rceil$ then $G$ is certainly pancyclic. El-Zahar [7] proved that if $G$ is of order $n=n_1+n_2$ with $n_1\geqslant 3$ and $n_2\geqslant 3$ and has minimum degree at least $\lceil n_1/2\rceil+\lceil n_2/2\rceil$, then $G$ contains two vertex-disjoint cycles of lengths $n_1$ and $n_2$, respectively. This result requires that the two vertex-disjoint cycles cover all the vertices of $G$. Amar [1] proved that if $G=(V_1,V_2;E)$ is a bipartite graph with $|V_1|=|V_2|=n\geqslant 4$ such that $d(x)+d(y)\geqslant n+2$ for all $x\in V_1$ and $y\in V_2$, then $G$ contains two vertex-disjoint cycles of lengths $2n_1$ and $2n_2$, respectively for each integer partition $n=n_1+n_2$ with $n_1\geqslant 2$ and $n_2\geqslant 2$. In [11], we generalized El-Zahar's result and proved the following two theorems.

**Theorem A** (Wang [11]). *Let $G$ be a graph of order $n\geqslant 6$ with minimum degree at least $\lceil (n+1)/2\rceil$. Then, for any two integers $s$ and $t$ with $s\geqslant 3$, $t\geqslant 3$ and $s+t\leqslant n$, $G$*

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*contains two vertex-disjoint cycles of lengths s and t, respectively, unless that n, s and t are odd and G is isomorphic to* $K_{(n-1)/2,(n-1)/2}+K_1$.

**Theorem B** (Wang [11]). *Let G be a graph of order* $n\geqslant 8$ *with n even and minimum degree at least* $n/2$. *Then, for any two even integers s and t with* $s\geqslant 4$, $t\geqslant 4$ *and* $s+t\leqslant n$, *G contains two vertex-disjoint cycles of lengths s and t, respectively.*

In this paper, we will prove the following:

**Theorem C.** *Let* $G=(V_1,V_2;E)$ *be a bipartite graph with* $|V_1|=|V_2|=n\geqslant 4$. *Suppose that* $d(x)+d(y)\geqslant n+2$ *for all* $x\in V_1$ *and* $y\in V_2$. *Then G contains two vertex-disjoint cycles of lengths* $2s$ *and* $2t$, *respectively, for any two positive integers s and t with* $s\geqslant 2$, $t\geqslant 2$ *and* $s+t\leqslant n$.

As mentioned in [1], the degree condition in Theorem C is sharp. Our method will also offer a shorter proof of Amar's result than that in [1], which will be accomplished within Case I of Section 3.

For convenience, we mention some terminology and notation. Let $G$ be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of $G$, respectively. We denote $|E(G)|$ by $e(G)$. For a vertex $u\in V(G)$ and a subgraph $H$ of $G$, we define $d(u,H)=|N(u)\cap V(H)|$. Hence $d(u,G)=d(u)$, the degree of $u$ in $G$. For a subset $U\subseteq V(G)$, $G[U]$ is the subgraph of $G$ induced by $U$, and $N(U)$ is $\bigcup_{u\in U}N(u)$. A graph is said to be traceable if it contains a hamiltonian path. A graph is called hamiltonian if it contains a hamiltonian cycle. For two vertex-disjoint subgraphs $G_1$ and $G_2$, $e(G_1,G_2)$ is the number of edges of $G$ between $G_1$ and $G_2$. Similarly, we define $e(X,Y)$ for two disjoint subsets $X$ and $Y$ of $V(G)$. For any two vertices $x$ and $y$ of $G$, we define $\mu(xy)=1$ if $xy$ is an edge of $G$ and $\mu(xy)=0$ otherwise.

To conclude our introduction, we propose the following conjecture.

**Conjecture D.** Let $G=(V_1,V_2;E)$ be a bipartite graph with $|V_1|=|V_2|=n\geqslant 2$. Suppose that $d(x)+d(y)\geqslant n+2$ for all $x\in V_1$ and $y\in V_2$. If $H=(U_1,U_2;F)$ is a bipartite graph with $|U_1|=|U_2|=n$ and $\Delta(H)\leqslant 2$, then $G$ contains a subgraph isomorphic to $H$.

## 2. Lemmas

In the following, $G=(V_1,V_2;E)$ is a bipartite graph with $|V_1|=|V_2|=n\geqslant 2$.

**Lemma 2.1.** *Let* $P=x_1y_1\ldots x_ky_k$ *be a path of G. Let* $y\in V(G)$ *be a vertex not on P such that* $\{x_1,y\}\not\subseteq V_i$ *for each* $i\in\{1,2\}$. *Suppose that* $d(x_1,P)+d(y,P)\geqslant k+1$. *Then G has a path* $P'$ *from y to* $y_k$ *such that* $V(P')=V(P)\cup\{y\}$.

**Proof.** The condition implies that $\{x_i y, x_1 y_i\} \subseteq E$ for some $i \in \{1,2,\ldots,k\}$. Then the path $P' = yx_i y_{i-1} \ldots x_2 y_1 x_1 y_i x_{i+1} y_{i+1} \ldots x_k y_k$ satisfies the requirement. $\square$

**Lemma 2.2** (Wang et al. [10]). *Let $P = x_1 y_1 \ldots x_k y_k$ be a path of $G$. Let $x \in V_1$ and $y \in V_2$ be vertices not on $P$. Then the following two statements hold*:

(a) *If $d(x,P) + d(y,P) \geqslant k + 2 - \mu(xy)$, then $G$ contains a path $P'$ from $x_1$ to $y_k$ such that $V(P') = V(P) \cup \{x, y\}$.*

(b) *If $d(x,P) + d(y,P) \geqslant k + 1 - \mu(xy)$, then $G$ contains a path $P'$ such that $V(P') = V(P) \cup \{x, y\}$.*

**Lemma 2.3** (Bondy and Chvátal [4]). *The following three statements hold*:

(a) *Let $P = x_1 y_1 \ldots x_k y_k$ be a path of $G$ with $k \geqslant 2$. If $d(x_1,P) + d(y_k,P) \geqslant k + 1$, then $G$ has a cycle $C$ such that $V(C) = V(P)$.*

(b) *If $d(x) + d(y) \geqslant n + 1$ for any two non-adjacent vertices $x$ and $y$ with $x \in V_1$ and $y \in V_2$, then $G$ is hamiltonian.*

(c) *If $d(x) + d(y) \geqslant n + 2$ for any two non-adjacent vertices $x$ and $y$ with $x \in V_1$ and $y \in V_2$, then $G$ is hamiltonian connected.*

**Lemma 2.4.** *Suppose that $d(x) + d(y) \geqslant n + 2$ for each pair of non-adjacent vertices $x \in V_1$ and $y \in V_2$. Then the following two statements hold*:

(a) *If $n \geqslant 3$, then $G - x - y$ is hamiltonian for all $x \in V_1$ and $y \in V_2$.*

(b) *For any three distinct vertices $x \in V_1$, $y_1 \in V_2$ and $y_2 \in V_2$, $G - x$ has a hamiltonian path from $y_1$ to $y_2$.*

**Proof.** Clearly, (a) follows from Lemma 2.3(b). We show (b) as follows. By (a), $G - x - y_2$ has a hamiltonian path $P = w_1 z_1 \ldots w_{n-1} z_{n-1}$ with $w_1 = y_1$. If $z_{n-1} y_2 \in E$, we are done. For otherwise, $d(z_{n-1},P) + d(y_2,P) \geqslant n + 2 - 1 = n + 1$, and so by Lemma 2.1, $(b)$ holds. $\square$

**Lemma 2.5** (Wang et al. [10]). *Suppose that $G$ has a hamiltonian path and for any two endvertices $u$ and $v$ of a hamiltonian path of $G$, $d(u) + d(v) \geqslant m$ holds, where $m$ is an integer greater than $n$. Then for every $x \in V_1$ and every $y \in V_2$, $d(x) + d(y) \geqslant m$.*

Let $V_1 = \{x_1,\ldots,x_n\}$ and $V_2 = \{y_1,\ldots,y_n\}$ be such that $d(x_1) \leqslant \cdots \leqslant d(x_n)$ and $d(y_1) \leqslant \cdots \leqslant d(y_n)$. Schmeichel and Mitchem [8] proved that if $n > 3$ and $d(x_k) \leqslant k$ implies $d(y_{n-k}) \geqslant n - k + 1$ for each $1 \leqslant k < n$, then $G$ contains a cycle of length $2l$ for each $l \in \{2,3,\ldots,n\}$. As a corollary, we have the following.

**Lemma 2.6.** *If $d(x) + d(y) \geqslant n + 1$ for each $x \in V_1$ and $y \in V_2$, then $G$ contains a cycle of length $2k$ for each $k \in \{2,3,\ldots,n\}$, unless $n = 3$ and $G$ is a cycle of length 6.*

**Lemma 2.7.** *Let $C$ be a cycle of length $2k$ in $G$. Let $xy$ be an edge in $G - V(C)$ such that $d(x,C)+d(y,C)\geqslant k$. If $C+x+y$ is not hamiltonian, then either $d(x,C)=0$ or $d(y,C)=0$.*

**Proof.** It is easy to see that if $d(x,C) > 0$ and $d(y,C) > 0$, then $C$ has an edge $uv$ such that $\{uy, vx\} \subseteq E$ and so $C + x + y$ is hamiltonian. □

## 3. Proof of Theorem C

We will use a well known result of Bondy and Chvátal [4]. For a bipartite graph $H = (X, Y; E)$ with $|X| = |Y| = n\geqslant 2$, the bipartite closure of $H$ is a bipartite graph obtained from $H$ by recursively joining pairs of nonadjacent vertices $x \in X$ and $y \in Y$ with the sum of their degrees at least $n + 1$. It is shown in [4] that $H$ is hamiltonian if and only if the bipartite closure of $H$ is hamiltonian. The following lemma is part of the proof of the theorem.

**Lemma 3.1.** *Let $p, q$ and $k$ be three positive integers with $k \geqslant p$ and $k + q < n$. Let $H$ be an induced traceable subgraph of order $2k$ in $G$. Let $F$ be an induced traceable subgraph of order $2q$ in $G - V(H)$. Let $z_1$ and $z_2$ be two vertices from $G - V(H \cup F)$ with $z_1 \in V_1$ and $z_2 \in V_2$. Set $D = G - V(H \cup F) - \{z_1, z_2\}$. Suppose the following hold*:

(a) *$d(u,H) + d(v,H)\geqslant k + 2$ for all $u \in V_1 \cap V(H)$ and $v \in V_2 \cap V(H)$*;
(b) *$d(x,F) + d(y,F)\geqslant q + 2$ for all $x \in V_1 \cap V(F)$ and $y \in V_2 \cap V(F)$*;
(c) *$D$ has a perfect matching*;
(d) *$d(z_1,H) + d(z_2,H)\geqslant k + 2$ and $d(z_1,F) + d(z_2,F)\geqslant 1$.*

*Then $G$ contains two vertex-disjoint cycles of lengths $2p$ and $2(q+1)$, respectively.*

**Proof of Lemma 3.1.** On the contrary, suppose the lemma false. Set $r = n - q$ and $H' = H \cup D$. By Lemma 2.6, $H$, consequently, $H'$, has a cycle of length $2p$. Let $(A, B)$ and $(X, Y)$ be the bipartitions of $F$ and $H'$, respectively, such that $A \cup X = V_1 - \{z_1\}$ and $B \cup Y = V_2 - \{z_2\}$. Let $Y_1 = N(A) \cap Y$, $X_1 = N(Y_1) \cap X$, and $X_2 = X - X_1$. Clearly, $|X_1| \geqslant |Y_1|$, and so $|X_2| \leqslant r - 1 - |Y_1|$. We break into the following two cases.

*Case* 1. For each $x \in X$ and $y \in Y$, $H' - x - y + z_1 + z_2$ contains a cycle of length $2p$.

Then $F + x + y$ is not hamiltonian for all $x \in X$ and $y \in Y$. By Lemma 2.3(c), $F$ is hamiltonian connected. Thus we see that for each edge $xy$ of $H'$, either $d(x,F) = 0$ or $d(y,F) = 0$. Furthermore, we see that for each $x \in X \cup \{z_1\}$ and $y \in Y \cup \{z_2\}$, $V(F)$ induces a complete bipartite graph in the bipartite closure of $F + x + y$, and therefore either $d(x,F) \leqslant 1$ or $d(y,F) \leqslant 1$. W.l.o.g., we may assume that

$$d(y,F) \leqslant 1 \quad \text{for all } y \in Y_1. \tag{1}$$

Clearly $e(X_1,F)=0$ and $d(z_1,F)+d(z_2,F)\leqslant q+1$. Thus

$$e(F,H')\leqslant|Y_1|+|X_2|q\leqslant(r-1)q-(q-1)|Y_1|.$$

On the other hand, we have

$$\begin{aligned} e(F,H') &\geqslant \sum_{u\in V(F)} d(u)-2e(F)-d(z_1,F)-d(z_2,F) \\ &\geqslant (r+q+2)q-2q^2-(q+1)=(r-q+1)q-1. \end{aligned}$$

Therefore $(r-1)q-(q-1)|Y_1|\geqslant(r-q+1)q-1$. This yields that $(q-1)^2\geqslant(q-1)|Y_1|$. By the assumption on the degrees of $F$, we see that $q\geqslant 2$, and so $|Y_1|\leqslant q-1$. By (1), $e(A,Y)=e(Y,A)\leqslant|Y_1|<|A|$. Then there exists $u\in A$ such that $d(u,Y)=0$. Let $w\in Y$. Then $d(u,F+z_1+z_2)+d(w,F+z_1+z_2)\leqslant q+2+d(w,F)$, and so $d(u,H')+d(w,H')\geqslant r-d(w,F)$. As $d(u,Y)=0$, we must have that $X\subseteq N(w)$ and $d(w,F)=1$. Thus $X_1=X$ and so $e(X,F)=0$. Then $d(x,G)\leqslant n-q$, $d(v,G)\leqslant q+1$ and so $d(x,G)+d(v,G)\leqslant n+1$ for each $x\in X$ and $v\in B$, a contradiction.

*Case* 2. For some $x_0\in X$ and $y_0\in Y$, $H'-x_0-y_0+z_1+z_2$ contains no cycle of length $2p$.

By Lemma 2.6, $H$ contains a cycle of length $2p$ if $\{x_0,y_0\}\cap V(H)=\emptyset$, a contradiction. W.l.o.g., say $x_0\in V(H)$. Let $y_0'\in V(H)$ be such that $y_0'=y_0$ if $y_0\in V(H)$. Then $d(x,H-x_0-y_0')+d(y,H-x_0-y_0')\geqslant k$ for all $x\in X\cap V(H)$ and $y\in Y\cap V(H)$ with $x\neq x_0$ and $y\neq y_0'$. Suppose $p<k$. By Lemma 2.6, $H-x_0-y_0'$ is a cycle of length 6. Therefore $p=2$. As $d(z_1,H)+d(z_2,H)\geqslant k+2$, $d(z_i,H-x_0-y_0')\geqslant 2$ for some $i\in\{1,2\}$. Clearly, $H-x_0-y_0'+z_i$ contains a cycle of length 4, a contradiction. Hence $p=k$. If $y_0\notin V(H)$, then by Lemma 2.4(b), $H-x_0+z_1$ is hamiltonian as $d(z_1,H)\geqslant 2$. Hence $y_0\in V(H)$. By Lemma 2.4(a), $H-x_0-y_0$ is hamiltonian or isomorphic to $K_2$. By Lemma 2.2(a), we must have that $d(z_1,H-x_0-y_0)+d(z_2,H-x_0-y_0)\leqslant k$. It follows that $\{x_0z_2,y_0z_1\}\subseteq E$. Let $Y_2=N_G(z_1)\cap Y$. Note that $y_0\in Y_2$. W.l.o.g., say $d(z_1,F)\geqslant 1$. Note that $d(z_2,H)\geqslant 2$. If there exists $w\in Y_1\cap Y_2$, we see that $F+z_1+w$ is hamiltonian as $F$ is hamiltonian connected, and by Lemma 2.4(b), $H-w+z_2$ is hamiltonian, a contradiction. Therefore $Y_1\cap Y_2=\emptyset$. Then we see that for each $x\in X$ and $y\in Y-Y_2$, $H-x-y+z_1+z_2$ is hamiltonian, and therefore $F+x+y$ is not hamiltonian. Consequently, $e(X_1,B)=0$.

First, suppose that $d(y,F)\leqslant 1$ for all $y\in Y_1$. Then as in Case 1, we can readily show that $|Y_1|\leqslant q-1$. For each $u\in A$, we must have that $d(u,H')\geqslant 1$ as $d(u)+d(y_0)\geqslant n+2$. It follows that $|Y_1|\geqslant q$, a contradiction.

Therefore $d(y',F)\geqslant 2$ for some $y'\in Y_1$. Then we must have that $d(x,F)\leqslant 1$ for all $x\in X$. Let $X_1'=N(B)\cap X$ and $Y_1'=N(X_1')\cap Y$. Then $X_1\cap X_1'=\emptyset$ and $Y_1'\cap Y_1=\emptyset$. Hence $e(Y_1',F)=0$. As in Case 1, we can readily show that $|X_1'|\leqslant q-1$. Let $x_1\in X_1$. Then we see that for each $v\in B$, $d(v,H')\geqslant 1$ as $d(x_1)+d(v)\geqslant n+2$. It follows that $|X_1'|\geqslant q$, a contradiction. This proves the lemma. □

Let $G=(V_1,V_2;E)$ be a bipartite graph with $|V_1|=|V_2|=n\geqslant 4$ such that $d(x)+d(y)\geqslant n+2$ for all $x\in V_1$ and $y\in V_2$. Suppose, for a contradiction, that

$G$ does not contain two vertex-disjoint cycles of lengths $2s$ and $2t$, respectively, for some integers $s$ and $t$ with $s \geqslant 2$, $t \geqslant 2$ and $s+t \leqslant n$. Let $r = n-s$. By Lemma 2.3(b), $G$ is hamiltonian. Then we can choose two vertex-disjoint induced subgraphs of $G$, say $G_1 = (A, B; E_1)$ and $G_2 = (X, Y; E_2)$ with $A \cup X = V_1$, of orders $2s$ and $2r$, respectively, such that

$$\text{both } G_1 \text{ and } G_2 \text{ are traceable.} \tag{2}$$

Subject to (2), we may further choose $G_1$ and $G_2$ such that

$$e(G_1) + e(G_2) \text{ is maximum.} \tag{3}$$

**Claim 1.** *Let $u$ and $v$ be two endvertices of a hamiltonian path of $G_1$ and let $x$ and $y$ be two endvertices of a hamiltonian path of $G_2$. Suppose that $uy \in E$ and $vx \in E$. Then*

$$\begin{aligned} &d(u,G_1) + d(v,G_1) + d(x,G_2) + d(y,G_2) \\ &\quad \geqslant d(u,G_2) + d(v,G_2) + d(x,G_1) + d(y,G_1). \end{aligned} \tag{4}$$

**Proof.** By (3), we must have that $e(G_1)+e(G_2) \geqslant e(G_1 - u + x) + e(G_2 - x + u)$ and $e(G_1) + e(G_2) \geqslant e(G_1 - v + y) + e(G_2 - y + v)$. This implies (4). □

**Claim 2.** *Let $u$ and $v$ be two endvertices of a hamiltonian path of $G_1$ and let $x$ and $y$ be two endvertices of a hamiltonian path of $G_2$ such that $u \in V_1$ and $x \in V_1$. Let $G_1' = G_1 - u - v + x + y$ and $G_2' = G_2 - x - y + u + v$. If both $G_1'$ and $G_2'$ are traceable, then*

$$\begin{aligned} &d(u,G_1) + d(v,G_1) + d(x,G_2) + d(y,G_2) \\ &\quad \geqslant d(u,G_2) + d(v,G_2) + d(x,G_1) + d(y,G_1) \\ &\qquad -2(\mu(uy) + \mu(vx)) + 2(\mu(uv) + \mu(xy)). \end{aligned} \tag{5}$$

*In particular, if $d(u,G_2)+d(v,G_2) \geqslant r+2$ and $d(x,G_1)+d(y,G_1) \geqslant s+2$, then* (5) *holds.*

**Proof.** If $d(u,G_2) + d(v,G_2) \geqslant r + 2$ and $d(x,G_1) + d(y,G_1) \geqslant s + 2$, then, by Lemma 2.2(b), both $G_1'$ and $G_2'$ are traceable. As both $G_1'$ and $G_2'$ are traceable, we have, by (3), that $e(G_1') + e(G_2') \leqslant e(G_1) + e(G_2)$, which implies (5). □

**Claim 3.** *Let $\{i,j\} = \{1,2\}$. If $G_i$ has a hamiltonian path with two endvertices $u$ and $v$ such that $d(u,G_i) + d(v,G_i) \leqslant |V(G_i)|$, then for all $x \in V_1 \cap V(G_j)$ and $y \in V_2 \cap V(G_j)$, $d(x,G_j) + d(y,G_j) \geqslant |V(G_j)| + 2$ holds.*

**Proof.** On the contrary, suppose the claim false. W.l.o.g., say $i=1$ and $j=2$. By Lemma 2.5, $G_2$ has a hamiltonian path $x_1 y_1 \ldots x_r y_r$ such that $d(x_1,G_2) + d(y_r,G_2) \leqslant r + 1$. Therefore $d(x_1,G_1) + d(y_r,G_1) \geqslant s+1$. Let $a_1 b_1 \ldots a_s b_s$ be a hamiltonian path of $G_1$ such that $d(a_1,G_1) + d(b_s,G_1) \leqslant s$. Thus $d(a_1,G_2) + d(b_s,G_2) \geqslant r+2$. Say $\{a_1, x_1\} \subseteq V_1$. By

Claim 1, $\mu(a_1y_r)+\mu(b_sx_1)\leqslant 1$. Let $G_1'=G_1-a_1-b_s+x_1+y_r$ and $G_2'=G_2-x_1-y_r+a_1+b_s$. By Lemma 2.2(b), both $G_1'$ and $G_2'$ are traceable. By Claim 2, we must have that $\mu(a_1y_r)+\mu(b_sx_1)=1$, $\mu(a_1b_s)=0$ and $\mu(x_1y_r)=0$. W.l.o.g., say $b_sx_1\in E$.

Clearly, we have that either $d(x_1,G_1)+d(a_1,G_2)>d(x_1,G_2)+d(a_1,G_1)$, or $d(y_r,G_1)+d(b_s,G_2)>d(y_r,G_2)+d(b_s,G_1)$. W.l.o.g., say the former holds. Clearly, $G_1-a_1+x_1$ is traceable. By (2) and (3), $G_2-x_1+a_1$ is not traceable. Then $\mu(a_1y_1)=0$. Note that we already have $d(a_1,G_2)+d(b_s,G_2)\geqslant r+2$ in the above. By Lemma 2.1, $d(a_1,G_2)+d(y_r,G_2)=d(a_1,G_2-x_1-y_1)+d(y_r,G_2-x_1-y_1)\leqslant r-1$. Therefore $d(b_s,G_2)\geqslant d(y_r,G_2)+3$. It follows that $d(a_1,G_1)+d(y_r,G_1)\geqslant s+3$, and so $d(y_r,G_1)\geqslant d(b_s,G_1)+3$. Furthermore, by Lemma 2.1, $G_1-b_s+y_r$ is traceable. Obviously, $G_2-y_r+b_s$ is traceable. But we obtain that $e(G_1-b_s+y_r)+e(G_2-y_r+b_s)>e(G_1)+e(G_2)$, a contradiction. This proves the claim. □

We now break our proof into the following two cases. W.l.o.g., we may assume that if $t=r$ then $G_1$ is not hamiltonian.

*Case* I. $G_1$ is not hamiltonian.

Let $P_1=a_1b_1\ldots a_sb_s$ be a hamiltonian path of $G_1$ with $a_1\in A$. By Lemma 2.3(a) and Claim 3, we have

$$a_1b_s\notin E,\ d(a_1,G_1)+d(b_s,G_1)\leqslant s \quad\text{and}\quad d(a_1,G_2)+d(b_s,G_2)\geqslant r+2; \tag{6}$$

$$d(x,G_2)+d(y,G_2)\geqslant r+2 \quad \text{for all } x\in X \text{ and } y\in Y. \tag{7}$$

Let $F=G_1-a_1-b_s$. By Lemmas 3.1 and 2.5, $F$ has a hamiltonian path $L=u_1v_1\ldots u_{s-1}v_{s-1}$ with $u_1\in A$ such that

$$d(u_1,F)+d(v_{s-1},F)\leqslant s. \tag{8}$$

First, suppose that $d(u_1,G_2)+d(v_{s-1},G_2)\leqslant r$. Then by (8), we have that $d(u_1,G_1)+d(v_{s-1},G_1)=s+2$, $\{u_1b_s,v_{s-1}a_1\}\subseteq E$, and $d(u_1,G_2)+d(v_{s-1},G_2)=r$. With (6), we see that either $N(a_1,G_2)\cap N(u_1,G_2)\neq\emptyset$ or $N(b_s,G_2)\cap N(v_{s-1},G_2)\neq\emptyset$. Say the former holds, and let $y'\in N(a_1,G_2)\cap N(u_1,G_2)$. Then $G_1-b_s+y'$ is hamiltonian. By (6), $d(b_s,G_2)\geqslant 2$. By Lemma 2.4(b), $G_2-y'+b_s$ is hamiltonian. Hence $t<r$. Then $G_2-y'-x$ does not contain a cycle of length $2t$ for any $x\in X$. By Lemma 2.6 and (7), $G_2-y'-x$ is a cycle of length 6 for all $x\in X$. This is impossible by (7).

Next, suppose $d(u_1,G_2)+d(v_{s-1},G_2)\geqslant r+1$. This implies that $G_2$ has an edge $xy$ such that $\{u_1y,v_{s-1}x\}\subseteq E$. Then $F+x+y$ is hamiltonian and so $G_2-x-y+a_1+b_s$ does not contain a cycle of length $2t$. First, suppose $t<r$. By Lemma 2.6 and (7), $G_2-x-y$ is a cycle of length 6, and so $t=2$. With (6), we see that $G_2-x-y+a_1+b_s$ contains a cycle of length 4, a contradiction. Therefore $t=r$. As $G_2-x-y$ is hamiltonian or isomorphic to $K_2$ by Lemma 2.4(a), we see that $d(a_1,G_2-x-y)+d(b_s,G_2-x-y)\leqslant r$ by Lemma 2.2(a). Hence equality holds in (6) and $\{a_1y,b_sx\}\subseteq E$. By (6), $d(a_1,G_2)\geqslant 2$ and $d(b_s,G_2)\geqslant 2$. By Lemma 2.4(b), both $G_2-x+a_1$ and $G_2-y+b_s$ are hamiltonian. Hence neither $F+b_s+x$ nor

$F + a_1 + y$ is hamiltonian. In particular, $a_1 v_{s-1} \notin E$ and $b_s u_1 \notin E$. We also have that $d(a_1, F - u_1 - v_{s-1}) + d(b_s, F - u_1 - v_{s-1}) = s$ since equality holds in (6). By Lemma 2.2(a), this implies that $G_1$ has a hamiltonian path from $u_1$ to $v_{s-1}$. Then we may repeat the above argument with $u_1$ and $v_{s-1}$ playing the role of $a_1$ and $b_s$ to show that $d(u_1, G_1) + d(v_{s-1}, G_1) = s$. It follows that either $d(a_1, F) + d(v_{s-1}, F) \geqslant s$ or $d(b_s, F) + d(u_1, F) \geqslant s$. W.l.o.g., say the former holds. By Lemma 2.1, $F + a_1$ has a hamiltonian path from $a_1$ to $u_1$. Therefore $F + a_1 + y$ is hamiltonian, a contradiction.

**Remark.** The following argument in Case II(a) shares much in common with the argument in Case I. We will point out below where the similarities occur in order to avoid some unnecessary duplication.

*Case* II. $G_1$ is hamiltonian.

By the choice of $G_1$ and $G_2$, we have that $t < k$. We break into the following two cases.

Case II(a). $G_2$ has a hamiltonian path $P_2 = x_1 y_1 \ldots x_r y_r$ with $x_1 \in X$ such that $d(x_1, G_2) + d(y_r, G_2) \leqslant r$.

With Claim 3, we obtain

$$d(x_1, G_1) + d(y_r, G_1) \geqslant s + 2; \tag{9}$$

$$d(u, G_1) + d(v, G_1) \geqslant s + 2 \quad \text{for all } u \in A \text{ and } v \in B. \tag{10}$$

Let $F = G[\{y_1, x_2, \ldots, y_{t-1}, x_t\}]$ and $D = G[\{y_t, x_{t+1}, \ldots, y_{r-1}, x_r\}]$. By Lemma 3.1, $F$ has a hamiltonian path $R = u_1 v_1 \ldots u_{t-1} v_{t-1}$ with $u_1 \in X$ such that

$$d(u_1, F) + d(v_{t-1}, F) \leqslant t. \tag{11}$$

As $G_2$ does not contain a cycle of length $2t$ and $D$ has a perfect matching, we see that

$$d(u_1, D) + d(v_{t-1}, D) \leqslant r - t. \tag{12}$$

It follows that

$$d(u_1, G_2) + d(v_{t-1}, G_2) \leqslant r + 2, \tag{13}$$

$$d(u_1, G_1) + d(v_{t-1}, G_1) \geqslant s. \tag{14}$$

Furthermore, if equality holds in (13) or (14), then $\{u_1 y_r, v_{t-1} x_1\} \subseteq E$.

First, suppose that $d(u_1, G_1) + d(v_{t-1}, G_1) = s$. Then we have that $\{x_1 v_{t-1}, u_1 y_r\} \subseteq E$. As in the third paragraph of Case I, we readily see that either $N(u_1, G_1) \cap N(x_1, G_1) \neq \emptyset$ or $N(v_{t-1}, G_1) \cap N(y_r, G_1) \neq \emptyset$. Then it follows that $G$ contains two required cycles, a contradiction.

Next, suppose that $d(u_1, G_1) + d(v_{t-1}, G_1) \geqslant s + 1$. This implies that $\{u_1 z, v_{t-1} w\} \subseteq E$ for some edge $wz$ of $G_1$. Hence $F + w + z$ is hamiltonian. As in the fourth paragraph of Case I, we readily deduce that $\{x_1 z, y_r w\} \subseteq E$. As $G_1 - w + x_1$ and $G_1 - z + y_r$ are hamiltonian, both $F + z + x_1$ and $F + w + y_r$ are not hamiltonian. As $d(x_1, G_1) + d(y_r, G_1) = s + 2$, we obtain that $d(x_1, G_2) + d(y_r, G_2) = r$. Furthermore, if $x_1 y_r \in E$ then $G_1 - w - z$

is hamiltonian or isomorphic to $K_2$ by Lemma 2.4(a) and so $G_1 - w - z + x_1 + y_r$ is hamiltonian by Lemma 2.2(a), a contradiction. So $x_1y_r \notin E$. This allows us to deduce a contradiction in the following.

As $d(x_1, G_2) + d(y_r, G_2) = r$, we may assume w.l.o.g. that $d(x_1, G_2 - x_1 - y_r) \geqslant r/2$. If $G_2 - x_1 - y_r$ is hamiltonian, then we readily see that $G_2 - y_r$ contains a cycle of length $2t$, a contradiction. Hence $G_2 - x_1 - y_r$ is not hamiltonian. By Lemma 2.3(a), $d(y_1, G_2 - x_1 - y_r) + d(x_r, G_2 - x_1 - y_r) \leqslant r - 1$. Hence $d(y_1, G_2) + d(x_r, G_2) \leqslant r + 1$. Thus $\sum_{x \in S} d(x, G_2) \leqslant 2r + 1$ where $S = \{x_1, y_1, x_r, y_r\}$. This implies that either $d(x_1, G_2) + d(y_1, G_2) \leqslant r$ or $d(x_r, G_2) + d(y_r, G_2) \leqslant r$. W.l.o.g., say the former holds. Then we define $F' = G[\{x_2, y_2, \ldots, x_{t-1}, y_{t-1}\}]$ and repeat the above argument with $x_1$, $y_1$ and $F'$ playing the role of $x_1$, $y_r$ and $F$, respectively. It follows that $x_1y_1 \notin E$, a contradiction.

*Case* II(b). For any two endvertices $x$ and $y$ of a hamiltonian path of $G_2$, $d(x, G_2) + d(y, G_2) \geqslant r + 1$ holds.

By Lemmas 2.5 and 2.6, $G_2$ is a cycle of length 6, say $x_1y_1x_2y_2x_3y_3x_1$ with $x_1 \in X$. Then $t = 2$. Let $C = a_1b_1 \ldots a_sb_sa_1$ be a hamiltonian cycle of $G_1$ with $a_1 \in A$. As $d(x_1, G_1) + d(y_1, G_1) \geqslant s + 1$, there exists an edge of $C$, say $a_sb_s$, such that $\{x_1b_s, y_1a_s\} \subseteq E$.

First, suppose that $G_1' = G_1 - a_s - b_s$ is hamiltonian or isomorphic to $K_2$. In the former case, let $C'$ be a hamiltonian cycle of $G_1'$. Clearly, we have that $d(x, G_1') + d(y, G_1') \geqslant s - 1$ for all $xy \in E(G_2 - x_1 - y_1)$. As $a_sb_sx_1y_1a_s$ is a cycle of length 4 in $G$, we see that $G_1' + x + y$ is not hamiltonian for each $xy \in E(G_2 - x_1 - y_1)$. By Lemma 2.7, either $d(x, G_1') = 0$ or $d(y, G_1') = 0$ for each $xy \in E(G_2 - x_1 - y_1)$. W.l.o.g., say $d(x_3, G_1) = s$ and $d(y_2, G_1) = 1$ with $y_2a_s \in E$. Then $d(y_3, G_1) = 1$ with $y_3a_s \in E$. We now see that $G$ contains two required cycles, a contradiction.

Next, suppose that $G_1'$ is not hamiltonian. By Lemma 2.3(a), $d(a_1, G_1') + d(b_{s-1}, G_1') \leqslant s - 1$ and so $d(a_1, G_1) + d(b_{s-1}, G_1) \leqslant s + 1$. Thus $d(a_1, G_2) + d(b_{s-1}, G_2) \geqslant 4$. But for each $xy \in E(G_2 - x_1 - y_1)$, $d(a_1, xy) + d(b_{s-1}, xy) \leqslant 1$ for otherwise we obtain the two required cycles immediately. Hence $\{a_1y_1, b_{s-1}x_1\} \subseteq E$. This argument allows us to see that we can deduce $d(u, x_1y_1) + d(v, x_1y_1) = 2$ for each edge $uv$ along $C$. Hence $d(x_1, G_1) = d(y_1, G_1) = s$. Similarly, we must have that $d(x_i, G_1) = d(y_i, G_1) = s$ for each $i \in \{2, 3\}$. Obviously, $G$ contains the two required cycles, a contradiction. This proves the theorem.

## 4. For further reading

The following references are also of interest to the reader: [5] and [9].

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